Discipline: **Physics**

Subject: **Electromagnetic Theory**

Unit 15:

Lesson/ Module: **Theory of Relativity - II**

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Learning Objectives:

In this module we continue to study the special theory of relativity. From this module students may get to know about the following:

1. *The light cone and the proper time of an event, a Lorentz invariant quantity.*
2. *The law of addition of velocities in relativity, i.e., the relation between the velocities of a particle as viewed from two different frames of reference.*
3. *Introduction to four vectors – the velocity and energy-momentum four-vectors.*
4. *The celebrated Einstein's mass-energy relation.*
5. *A mathematical description of the properties of space-time in special theory of relativity and introduction to four-tensors.*

I4. Theory of Relativity - II

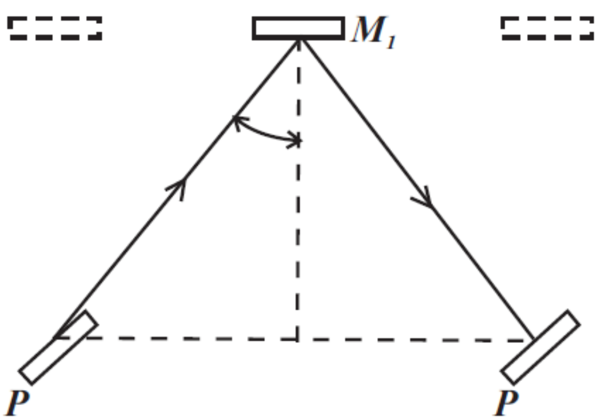
In the last module we had begun our study of the Einstein's special theory of relativity. We had described the experimental basis of the need for the special theory and introduced the two Einstein's basic postulates of special relativity. We had then "derived" the Lorentz transformations and briefly discussed some of its salient consequences. Finally we had briefly introduced the concept of *four vectors.*

In this module we continue with the theory of relativity, discuss some further consequences of Lorentz transformations, and study in detail the space-time structure of special relativity.

15.1 Light Cone and Proper Time

A useful concept in special relativity is that of *light cone* and *space-like* and *time-like* intervals between two events. Consider the figure shown below in **[SEE FIGURE 11.3 Jackson Edition 2]** which the time axis *ct* is vertical and the space axes are perpendicular to it. For simplicity we show only the *x*-axis. At time t=0, a particle is at the origin. Because no particle or physical system can have speed greater than the speed of light, the space-time domain can be divided into three regions by a cone, called the *light cone* whose surface is specified by the equation

$$x^2 + y^2 + z^2 - c^2t^2 = 0. \qquad (4)$$



In the diagram of course it appears as a pair of straight lines. Light signals emitted from the origin at $t = 0$ would travel along these lines in the figure. But any material system has a speed less than *c*, and hence, as time progresses, it would trace out a path, called its *world line*. This world line will lie inside the upper half cone. In fact, since the speed of the particle at any instant is given by the cotangent to this curve and speed of the particle cannot exceed *c*, at every instant the angle made by the tangent to the curve must be greater than 45^0. Since the path of the particle must lie in the upper half cone for $t > 0$, that region is called the *future.* Similarly, the particle must have reached the origin at $t = 0$ by some path which must lie in the lower half-cone called the *past*. The shaded region in the figure for which $x^2 + y^2 + z^2 > c^2t^2$ is called *elsewhere*. A system at *O* at t=0 can never reach or come from a point in the elsewhere region, because for that to happen its speed will have to exceed *c* at some time or the other.

This division of the space-time into past, future and elsewhere regions can be understood from the concept of *invariant separation* or *interval* between two events $(t_1, \vec{x}_1)$ and $(t_2, \vec{x}_2)$ is defined as

$$s_{12}^{\ 2} = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2 . \quad (5)$$

For two infinitesimally close events, the infinitesimal interval is defined by

$$ds^2 = c^2 dt^2 - d\vec{x}^2 . \quad (6)$$

In a different frame of reference, the coordinates of the two events would be $(t_1', \vec{x}_1')$ and $(t_2', \vec{x}_2')$, but the interval between the two would be same:

$$s_{12}'^2 = s_{12}^{\ 2} \quad (7)$$

For any two events, there are three possibilities: $s_{12}^{\ 2} > 0,\ s_{12}^{\ 2} < 0,\ s_{12}^{\ 2} = 0$.

$(i)\ s_{12}^{\ 2} > 0$**.** In this case the events are said to have a *time-like separation*. We can always find a frame of reference by a Lorentz transformation in which $\vec{x}_1' = \vec{x}_2'$. Then

$$s_{12}'^2 = c^2(t_1' - t_2')^2 > 0$$

In this frame the two events occur at the same point ($\vec{x}_1' = \vec{x}_2'$) but are always separated in time. For, if a frame of reference could be found in which the two events are simultaneous, then in that frame

$$s_{12}'^2 = -|\vec{x}_1 - \vec{x}_2|^2 < 0$$

which is not possible since $s_{12}'^2 = s_{12}^{\ 2}$, and $s_{12}^{\ 2} > 0$. In the figure for two such events if one point is at the origin the other is in the past or future region.

$(ii)\ s_{12}^{\ 2} < 0$. In this case the events are said to have a *space-like separation*. Now it is always possible to find a frame of reference in which $t_1' = t_2'$, since then

$$s_{12}'^2 = -|\vec{x}_1 - \vec{x}_2|^2 < 0 .$$

However, now it is not possible to find a frame in which $\vec{x}_1' = \vec{x}_2'$, for if such a frame could exist, then in that frame

$$s_{12}'^2 = c^2(t_1 - t_2)^2 > 0 .$$

This is not possible since $s_{12}'^2 = s_{12}^{\ 2}$, and $s_{12}^{\ 2} < 0$. Two such events can occur at the same time but never at the same point – the separation is space-like. In the figure for two such events if one point is at the origin the other has to be in the elsewhere region.

$(iii)\ s_{12}{}^2 = 0$. In this case the events are said to have a *light-like separation*. Two such events can be connected only by a light signal. If one point lies at the origin, the other must lie on the light cone.

The division of the separation between two events into three categories is *Loretz invariant*, i.e., it is independent of the frame of reference, since the separation depends on $s_{12}{}^2$, which has the same value in all frames; and thereby lies its importance. An event which is in the past (future) of another event in one frame is in the past (future) in *all frames*. Thus past (future) are *absolute past (future)*. If two events have a space-like separation in one frame then the separation is space-like in *all frames*. Two such events cannot be causally connected as all interactions propagate with speeds not exceeding the speed of light.

15.1.1 Proper time

Consider a particle moving with velocity $\vec{u}(t)$ in some inertial frame K. The square of the infinitesimal invariant interval *ds* is

$$ds^2 = c^2 dt^2 - d\vec{x}^2 = c^2 dt^2 (1-\beta^2); \qquad \beta = \frac{|\vec{u}|}{c}. \tag{8}$$

Consider the frame of reference in which the particle is instantaneously at rest. In this frame

$$d\vec{x}' = 0, \qquad dt'^2 = dt^2(1-\beta^2) = dt^2 / \gamma(t)^2,$$

or

$$d\tau \equiv dt' = \frac{dt}{\gamma(t)} \tag{9}$$

The time τ is called the *proper time of the particle*. It is the time as seen in the rest-frame of the system. It follows on integrating the above equation that a proper time interval (τ_2 - τ_1) - say time of travel of the particle in the rest frame - will appear in the moving frame K as time interval

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \gamma(\tau) d\tau . \tag{10}$$

These equations also express the phenomenon of *time dilatation*, applied to a system which may not be moving with uniform velocity. Time interval $d\tau$ in the rest frame appears as an interval $\gamma(\tau)d\tau > d\tau$ in the moving frame; in other words a moving clock runs slower than a stationary one.

15.2 Addition of velocities

The Lorentz transformations (1) or inverse transformation (2) can be used to obtain law of addition of velocities. Let a particle be moving with speed $\vec{u}'$ in the frame K', which itself is moving with a speed *v* in the *x* direction with respect to a frame K. Then from (2), for infinitesimal intervals we obtain

$$dx_0 = \gamma(dx_0' + \beta dx_1')\,, \quad dx_1 = \gamma(dx_1' + \beta dx_0'),\; dx_2 = dx_2'\,, \quad dx_3 = dx_3' \tag{11}$$

If the velocity of the particle in the K frame is denoted by $\vec{u}$, we have

$$\begin{aligned} u_1 &= c\frac{dx_1}{dx_0} = c\frac{dx_1' + \beta dx_0'}{dx_0' + \beta dx_1'} = \frac{u_1' + v}{1 + u_1' v / c^2}; \\ u_2 &= c\frac{dx_2}{dx_0} = c\frac{dx_2'}{\gamma(dx_0' + \beta dx_1')} = \frac{u_2'}{\gamma(1 + u_1' v / c^2)}; \\ u_3 &= c\frac{dx_3}{dx_0} = c\frac{dx_3'}{\gamma(dx_0' + \beta dx_1')} = \frac{u_3'}{\gamma(1 + u_1' v / c^2)}. \end{aligned} \tag{12}$$

We see that the component parallel to the relative speed of two frames transforms differently than the transverse components. If we denote the parallel and transverse components by subscripts $\|$ and $\perp$ respectively, then

$$\vec{u}'_{\|} = \frac{\vec{u}_{\|} + \vec{v}}{1 + \dfrac{\vec{v}.\vec{u}'}{c^2}}\,, \tag{13}$$

$$\vec{u}'_{\perp} = \frac{\vec{u}'_{\perp}}{\gamma(1 + \dfrac{\vec{v}.\vec{u}'}{c^2})}\,. \tag{14}$$

Taking the simple case of all velocities in the *x*-direction, we have

$$u = \frac{u' + v}{1 + \dfrac{v.u'}{c^2}} \tag{15}$$

Consider various cases of interest. If both the speeds u' and v are small compared to the speed of light, the denominator can be replaced by unity, and we recover the well-known Newtonian transformation law for velocities

$$u = u' + v$$

Einstein relativity approaches Newtonian relativity for “small” velocities.

Putting $u'=c$, $v=c$, or both equal to c, we obtain $u=c$. Thus c is indeed the upper limit for all speeds, and c cannot be achieved by any particle or system. For a beam of light produced by a source that is stationary in K', u' equals c, and so does u from above. Thus the speed of light is the same in all frames independent of the motion of the source. This of course was the starting point of the Einstein’s relativity; we simply recover the same result indicating the internal consistencies of the formalism.

15.3 Velocity and energy-momentum four vectors

As we have said earlier, we anticipate, in analogy with ordinary vectors, the existence of physical quantities that behave like space-time coordinates under Lorentz transformations; we called such quantities four vectors. The coordinate x_0 is the- component of the four vector, whereas (x_1, x_2, x_3) are the space components. If we look at the structure of equations (13) and (14), we realize that law of transformation of velocities is not that of the space component of a four vector. The reason is that in $\vec{u} = \frac{d\vec{x}}{dt}$, $d\vec{x}$ is the space component of a four vector but *dt* is not invariant under Lorentz transformation; it is time component of a four vector. If we have a related quantity which is invariant under Lorentz transformation, then $\vec{u}$ will transform like $d\vec{x}$. We have already found this quantity, viz., the proper time interval $d\tau$, which as we have already seen is a "Lorentz invariant". Thus dividing by $d\tau$ instead of *dt* we can construct a four vector from $(x_0, \vec{x})$:

$$U^0 = \frac{dx^0}{d\tau} = \frac{dx^0}{dt}\frac{dt}{d\tau} = \gamma c;;$$

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \gamma\vec{u}.$$

Together the two can be written as

$$U^\mu = \frac{dx^\mu}{d\tau} \tag{16}$$

15.3.1 Energy-momentum four vector

Let us try to form another four-vector starting with the velocity four-vector, viz.,

$$P^\mu = mU^\mu. \tag{17}$$

Analysing the space and time components of this four-vector, we have

$$\vec{P} = m\vec{U} = m\gamma\vec{u}; \qquad P_0 = mU_0 = m\gamma c.$$

The vector $\vec{P}$ is like the nonrelativistic momentum, except that the mass is multiplied by the factor γ. Thus we can regard $m\gamma$ as the effective mass of the particle:

$$m\gamma = \frac{m}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{18}$$

So the mass of a moving particle gets replaced by $\frac{m}{\sqrt{1-\frac{u^2}{c^2}}}$ - a moving particle gets heavier as its speed increases. The "ordinary" mass is usually written as m_0 and called the *rest mass* of the particle. The mass of the particle having velocity v is $m(v) = m_0\gamma$.

Notice that $m(v) \to \infty$ as $v \to c$. Thus a material particle moving with the speed of light would have infinite mass; it is another way of saying that no material particle can move with the speed of light.

15.4 Einstein's mass-energy relation

Let us now look at the time component of the four vector P^μ, $P_0 = m\gamma c$, in fact at cP_0, which has dimensions of energy, E. Expanding γ in powers of v^2/c^2, we have

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = (1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots)$$

so that

$$E = \gamma mc^2 = mc^2(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots) = mc^2 + \frac{1}{2}mv^2 + \ldots$$

The second term in this expansion is just the usual kinetic energy of a particle in Newtonian mechanics. The first term is a constant, independent of the velocity of the particle and present even when the particle is at rest. It is this identification that led Einstein to boldly suggest that a particle of mass m has energy

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad (19)$$

whether it is at rest or in motion. Thus a particle at rest has energy

$$E = mc^2 \qquad (20)$$

often called the *rest energy* of the particle. This is the celebrated Einstein's mass-energy equivalence relation. It says that mass and energy are equivalent and interchangeable.

15.5 Mathematical Properties of Space-Time in Special Relativity

We have introduced the concept of four vectors in a heuristic manner. We now develop the properties of *space-time manifold* in a more consistent manner. Three-dimensional rotations in classical and quantum mechanics can be discussed in terms of group of transformations of coordinates in space that leave the *norm* of the vector $\vec{x}$ invariant. In fact such transformations also include translations and reflections. In the special theory of relativity the relationship between coordinates in different inertial frames are provided by Lorentz transformations that leave the interval

$$s^2 = x_0^{\ 2} - x_1^{\ 2} - x_2^{\ 2} - x_3^{\ 2} \tag{21}$$

invariant. The group of transformations that leave s^2 invariant is called the *homogenous Lorentz group*. This includes the Lorentz transformations and also include the ordinary rotations, since within one given frame of reference rotations will leave not only the norm of $\vec{x}$ invariant but also the interval s^2.

That space and time are homogenous and isotropic is one of the most sacrosanct principles of nature. It means there is no preferred direction or position in space. It means the results of any experiment should be independent of the coordinates chosen. It further implies that the left hand side and the right hand side of an equation should transform in the same manner under rotations, so that an equation valid in one coordinate system will be valid in any other rotated system. The transformation of coordinates under rotations can be written as a linear transformation:

$$x_i' = \sum_{j=1}^{3} a_{ij} x_j, \quad i = 1,2,3. \tag{22}$$

Since the norm of $\vec{x}$ must be invariant, it leads to the following relation among the coefficients a_{ij}:

$$\sum_{k=1}^{3} a_{ik} a_{jk} = \delta_{ij} \tag{23}$$

Here δ_{ij} is the *kronecker delta symbol* and has the value one if the indices are equal and zero otherwise.

We next define a *vector* as any set of three physical quantities that behave under rotations like the coordinates x_i. Likewise we define a *tensor of rank 2* as any set of nine physical quantities that transform under rotations in the following manner:

$$T_{ij}' = \sum_{k=1}^{3} \sum_{l=1}^{3} a_{ik} a_{jl} T_{kl}, \quad i, j = 1,2,3. \tag{24}$$

One can extend this definition to tensors of higher ranks as well.

The notation can be much simplified by using *Einstein summation convention*. Under this convention:

(i) An index that appears once in an expression is a *free* index and represents a set of equations for its three values. Thus two free indeces will represent a set of nine equations and so on.

(ii) An index that appears twice in an expression is understood to be summed over.

(iii) No index can appear more than twice in an equation.

Needless to say, free indices must match in every term of an expression and one repeated index can be replaced by another. Thus the above equations can be written as

$$x_i' = a_{ij} x_j \tag{25}$$

$$a_{ik} a_{jk} = \delta_{ij} \tag{26}$$

$$T_{ij}' = a_{ik} a_{jl} T_{kl} . \tag{27}$$

After this brief digression to recall some of the well-known results from rotations in three-dimensional space, let us get back to our four dimensional space-time continuum.

From the first of the Einstein's postulates the mathematical equations expressing the laws of nature must be the same in all inertial frames of reference, i.e., they must be invariant in form under Lorentz transformations, or they must be *covariant*. In other words every term on the right hand or left hand side of an equation must transform in the same manner under Lorentz transformations. In analogy with the case of rotations in ordinary space, this implies that such equations must be expressed by relations among *four-scalars, four-vectors, four-tensors* etc. Just as ordinary scalars, vectors and tensors are defined by their transformation properties under rotations, four scalars, vectors and tensors etc. are defined by their properties under Lorentz transformations.

However, there is a difference also between the two cases. In the norm of the position vector all terms appear with the same (positive) sign, whereas in the *interval* which is the corresponding quantity in our four dimensional case, one of the terms appears with an opposite sign. This makes the space *non-Euclidean* with its accompanying complications.

To begin with we keep our discussion sufficiently general and do not restrict ourselves to a specific transformation law from the unprimed coordinates $(x^0, x^{1,} x^2, x^3)$ to primed coordinates $(x'^0, x'^{1,} x'^2, x'^3)$. We suppose that there is a well-defined law of transformation between the two:

$$x'^\alpha = x'^\alpha (x^0, x^1, x^2, x^3); \quad \alpha = 0,1,2,3. \tag{28}$$

This transformation is invertible; the Jacobian of the transformation is not equal to zero. Thus one can, in principle, solve these set of equations to obtain

$$x^\alpha = x^\alpha (x'^0, x'^1, x'^2, x'^3); \quad \alpha = 0,1,2,3. \tag{29}$$

Scalars, vectors and in general tensors of certain rank are defined by their transformation properties. A *(Lorentz) scalar*, or a tensor of rank zero is a single quantity whose value is not changed by the transformation. The interval ds^2 is obviously one such quantity. (So are the charge of a particle and the rest mass of a particle.)

For tensors of rank one, or *vectors*, two kinds have now to be distinguished. A *contravariant vector* A^α, is a set of four quantities which transform according to the rule

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta . \tag{30}$$

In this equation the derivatives are evaluated from equation (28). Just to remind you once, the repeated index β is summed over; the index α is free; thus the above equation represents a set of four equations:

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^0} A^0 + \frac{\partial x'^\alpha}{\partial x^1} A^1 + \frac{\partial x'^\alpha}{\partial x^2} A^2 + \frac{\partial x'^\alpha}{\partial x^3} A^3; \quad \alpha = 0,1,2,3. \tag{31}$$

In a contravariant vector, the indices appear as superscripts.

A covariant vector or tensor of rank one, is a set of four quantities which transform according to the rule

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta \tag{32}$$

or explicitly by

$$B'_\alpha = \frac{\partial x^0}{\partial x'^\alpha} B_0 + \frac{\partial x^1}{\partial x'^\alpha} B_1 + \frac{\partial x^2}{\partial x'^\alpha} B_2 + \frac{\partial x^3}{\partial x'^\alpha} B_3; \quad \alpha = 0,1,2,3 . \tag{33}$$

In the covariant case the derivatives are evaluated from equation (29) which is the inverse transformation of (28) and express the coordinates x^α in terms of x'^β. The indices in a covariant vector appear as subscripts.

If the law of transformation (28) is linear, then the components (x^0, x^1, x^2, x^3) do indeed form components of a contravarient four vector.

A generalization to tensors of higher order in now clear. A *contravariant tensor of rank two* $F^{\alpha\beta}$ is a set of sixteen quantities that transform according to

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta} . \tag{34}$$

A *covariant tensor of rank two* $G_{\alpha\beta}$ is a set of sixteen quantities that transform according to

$$G'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} G_{\gamma\delta} . \tag{35}$$

However, we can now have a *mixed tensor of rank two* $H'^\alpha{}_\beta$ which transforms according to

$$H'^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial x'_{\beta}} H^{\gamma}{}_{\delta} . \quad (36)$$

Generalization to even higher ranks of contravariant, covariant and mixed tensors is now straightforward.

The *inner or scalar product* is defined when one vector is contravariant and the other covariant. Thus if *A* is contravariant and *B* covariant, their scalar product is defined as

$$A.B \equiv A^{\alpha} B_{\alpha} \quad (37)$$

Just as in the case of three vectors, the scalar product of two vectors is a scalar, the inner product of two four vectors as defined above is also an invariant:

$$A'.B' = A'^{\alpha} B'_{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} B_{\gamma} \quad (38)$$

Now using the chain rule of partial differentiation

$$\frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \frac{\partial x^{\gamma}}{\partial x^{\beta}} = \delta^{\gamma}{}_{\beta} \quad (39)$$

Here $\delta^{\gamma}{}_{\beta}$ is the Kronecker delta function in four dimensions: unity if the two indices are same and zero otherwise. Thus

$$A'.B' = A^{\beta} B_{\gamma} \delta^{\gamma}{}_{\beta} = A^{\beta} B_{\beta} = A.B . \quad (40)$$

You can easily verify that a similar definition of inner product with both vectors either contravariant or covariant *do not* yield a scalar: $A'^{\alpha} B'^{\alpha} \neq A^{\alpha} B^{\alpha}$.

15.5.1 The Metric Tensor of Special Relativity

So far our discussion was general (and could be applied even to general relativity). We now specialize to the space-time of special relativity. In this case the invariant interval is given by equation (1). In differential form the invariant interval that defines the norm in our space is given by

$$(ds')^2 = (ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

If we write our *norm or metric* as

$$(ds)^2 = g_{\alpha\beta} (dx)^{\alpha} (dx)^{\beta} , \quad (41)$$

then for the space-time of special relativity the *metric tensor* is diagonal, and is given by

$$g_{00} = 1, \qquad g_{11} = g_{22} = g_{33} = -1. \tag{42}$$

We can define the corresponding *contravariant metric tensor* $g^{\alpha\beta} = g_{\alpha\beta}$ numerically. The contraction of the contravariant and the covariant metric tensors gives the Kronecker product in four dimensions:

$$g_{\alpha\gamma} g^{\gamma\beta} = \delta_{\alpha}^{\ \beta} . \tag{43}$$

Very often it is convenient to write $g_{\alpha\beta}$ in a matrix form:

$$g_{\alpha\beta} = g^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{44}$$

In matrix form the invariant length is

$$x^{\alpha} g_{\alpha\beta} x^{\beta} = \begin{bmatrix} x^0 & x^1 & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = (ds)^2$$

15.5.2 Raising and lowering of indices

The metric tensor $g_{\alpha\beta}$ and $g^{\alpha\beta}$ can be used for the operation of *lowering and raising indices* of a given tensorial quantity. For example given a contravariant vector A^{α} we use metric tensor to define the *corresponding* covariant vector A_{α} by

$$A_{\alpha} = g_{\alpha\beta} A^{\beta} \tag{45}$$

In particular

$$x_{\alpha} = g_{\alpha\beta} x^{\beta} \tag{46}$$

If the components of A^{α} are (A^0, A^1, A^2, A^3), then those of A_{α} are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} = \begin{bmatrix} A^0 \\ -A^1 \\ -A^2 \\ -A^3 \end{bmatrix}$$

In other words,

$$A_0 = A^0; \qquad A_1 = -A^1; \qquad A_2 = -A^2; \qquad A_3 = -A^3. \tag{47}$$

Thus with every "natural" contravariant four vector is associated a covariant four vector. The two have time components same but space components opposite in sign. Thus $x^\mu = (x^0, \vec{x})$ is a natural contravariant; its covariant "partner" is $x_\mu = (x^0, -\vec{x})$.

15.5.3 Contraction of indices

The inner or scalar product of two vectors is a special case of contraction of two indices. Using the metric tensor, this product can be put in different forms:

$$A.B = A^\alpha B_\alpha = g^{\alpha\beta} A_\beta B_\alpha = g_{\alpha\beta} A^\alpha B^\beta = B^0 A^0 - \vec{A}.\vec{B} \tag{48}$$

The contraction can be applied to any two indices, one of which is contravariant and the other covariant. For example, from the tensor $T^{\alpha\beta}$, we obtain the scalar

$$T^\alpha{}_\alpha = g_{\alpha\beta} g_{\alpha\gamma} T^{\beta\gamma}. \tag{49}$$

Whenever two indices are contracted, the rank of a tensor is reduced by two, since the number of free indices is reduced by two.

15.5.4 The differential operator

Consider now the partial derivative operator with respect to x^α and x_α. The transformation properties of these operators can be established by using the rules of implicit differentiation. For example we have

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta} = \frac{\partial x^0}{\partial x'^\alpha}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x'^\alpha}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^\alpha}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^\alpha}\frac{\partial}{\partial x^3}. \tag{50}$$

By comparison with equation (32) for transformation of covariant vectors, we see that differentiation with respect to a contravariant component of the coordinate vector transforms as a component of a covariant vector. Thus x'^α is contravariant but $\dfrac{\partial}{\partial x'^\alpha}$ is a *covariant operator.* Similarly differentiation with respect to a covariant component gives a *contravariant operator*:

$$\frac{\partial}{\partial x^\gamma} = \frac{\partial x_\alpha}{\partial x^\gamma}\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial x^\gamma}(g_{\alpha\beta} x^\beta)\frac{\partial}{\partial x_\alpha} = g_{\alpha\beta}\delta^\beta{}_\gamma \frac{\partial}{\partial x_\alpha} = g_{\alpha\gamma}\frac{\partial}{\partial x_\alpha}$$

Alternatively, operating on both sides of the above equation with $g'^{\beta\gamma}$ (while using the prime coordinates) yields a gradient operator that transforms as a contravariant tensor, since, using equation (14) [$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}}\frac{\partial x'^{\beta}}{\partial x^{\delta}}F^{\gamma\delta}$] and (50) above, we have

$$\frac{\partial}{\partial x'_{\alpha}} = g'^{\alpha\gamma}\frac{\partial}{\partial x'^{\gamma}} = (g^{\mu\nu}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}.\frac{\partial x'^{\gamma}}{\partial x^{\nu}})(\frac{\partial x^{\beta}}{x'^{\gamma}}.\frac{\partial}{\partial x^{\beta}})$$

$$= g^{\mu\nu}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}(\frac{\partial x'^{\gamma}}{\partial x^{\nu}}.\frac{\partial x^{\beta}}{\partial x'^{\gamma}})\frac{\partial}{\partial x^{\beta}} = g^{\mu\nu}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\delta^{\beta}{}_{\nu}\frac{\partial}{\partial x^{\beta}} = g^{\mu\nu}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\beta}}$$

$$= \frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial x_{\mu}}.$$

We therefore employ the notation

$$\partial^{\alpha} \equiv \frac{\partial}{\partial x_{\alpha}} = (\frac{\partial}{\partial x^{0}}, -\vec{\nabla}) \qquad (51)$$

$$\partial_{\alpha} \equiv \frac{\partial}{\partial x^{\alpha}} = (\frac{\partial}{\partial x^{0}}, \vec{\nabla}) \qquad (52)$$

The quantity ∂^{α} is the four-dimensional generalization of gradient. Similarly we have the *four-divergence* of a four-vector:

$$\partial^{\alpha}A_{\alpha} = \partial_{\alpha}A^{\alpha} = \frac{\partial A^{0}}{\partial x^{0}} + \vec{\nabla}.\vec{A} \qquad (53)$$

Finally, the invariant four-dimensional *Laplacian operator* is defined by contraction of the differential operator:

$$\Box = \partial^{\alpha}\partial_{\alpha} = \frac{\partial^{2}}{\partial x^{0^{2}}} - \vec{\nabla}^{2} \qquad (54)$$

15.5.5 Matrix representation of Lorentz transformations

The specific Lorentz transformation between two frames K and K', when K' is moving along the positive *x*-axis is given by

$$x' = \gamma(x - vt)$$
$$t' = \gamma(-\frac{v}{c^2}x + t)$$
$$y' = y$$
$$z' = z$$

In our present notation this takes the form

$$x'^0 = \gamma(x^0 - \beta x^1)$$
$$x'^1 = \gamma(-\beta x^0 + x^1)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

or if we represent the Lorentz transformation by Λ,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu ,$$

the matrix Λ is given by

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (55)$$

The inverse transformation from primed to unprimed coordinates is given by the inverse matrix

$$x^\mu = (\Lambda^{-1})^\mu{}_\nu x'^\nu ,$$

$$\Lambda^{-1} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (56)$$

Summary

1. *In this module the study of the special theory of relativity was continued.*
2. *The light cone and the proper time of an event are discussed. The proper time of an event is a Lorenz invariant quantity unlike the usual time which is different in different frames of reference.*
3. *The law of addition of velocities, i.e., the relation between the velocities of a particle as viewed from two different frames of reference is derived from the Lorentz transformations. The law of addition that we get differs profoundly from the usual law of addition of velocities which follows from the Galilean transformations. The law clearly demonstrates that the velocity of light is the upper limit to the velocity achievable by any material particle.*
4. *The concept of four-vectors is introduced and the velocity and energy-momentum four-vectors are discussed.*
5. *The celebrated Einstein's mass-energy relation is derived.*
6. *A mathematical description of the properties of space-time in special theory of relativity is given. This leads to two kinds of four-vectors, the contravariant and the covariant, because of the peculiar structure of space-time in Einstein's relativity. The idea of four vectors is extended to four-tensors much on the line of ordinary vectors but with some clear difference.*